Number 6 - May 2021 Problems

22 June 2021

Problems

Problem 35A. Proposed by DC

In trapezoid ABCD, the bases are AB=7 cm and CD=3 cm. The circle with the origin at A and radius AD intersects diagonal AC at M and N. Calculate the value of the product $CM \times CN$.

Problem 41A. Proposed by Alexander Monteith-Pistor

Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 1$ and

$$f(p^k m) = \sum_{i=0}^{k-1} f(p^i m)$$

for all $p, k, m \in \mathbb{N}$ where p is a prime which does not divide m.

Solution Problem 41A

Let $n \geq 2$ be an integer with unique factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ (where $p_1, ..., p_k$ are primes and $\alpha_1, ..., \alpha_k$ positive integers). We prove, by induction on $\sum_{i=1}^{k} \alpha_i$, that

$$f(n) = 2^{\sum_{i=1}^{k}(\alpha_i - 1)}$$

Base case: $\sum_{i=1}^{k} \alpha_i = 1$. Then $n = p$ for some prime p. Thus,

$$f(p) = f(p^1 \cdot 1) = \sum_{i=0}^{0} f(p^i \cdot 1) = f(1) = 1$$

Inductive Hypothesis: assume that $n > 2$ and for all $n' < n$

$$f(n') = 2^{\sum_{i=1}^{k}(\beta_i - 1)}$$

where $n' = \prod_{i=1}^{k} q_i^{\beta_i}$. Then, letting $m = \prod_{i=1}^{k-1} p_i^{\alpha_i}$

$$f(n) = f(p_k^{\alpha_k} \cdot m) = \sum_{i=0}^{\alpha_k - 1} f(p_k^i \cdot m)$$

Applying the inductive hypothesis,

$$\begin{aligned} f(n) &= 2^{\sum_{j=1}^{k-1}\alpha_j} + \sum_{i=1}^{\alpha_k-1} 2^{i-1}\cdot 2^{\sum_{j=1}^{k-1}(\alpha_j-1)} \\ &= \left(1+\sum_{i=0}^{\alpha_k-2} 2^i\right)\cdot 2^{\sum_{j=1}^{k-1}(\alpha_j-1)} \\ &= 2^{\sum_{j=1}^{k}(\alpha_j-1)} \end{aligned}$$

which concludes the proof by induction.

Problem 42A. Proposed by Vedaant Srivastava

Given positive reals a, b, c, prove that

$$\frac{a(a^3+1)}{2b+6c}+\frac{b(b^3+1)}{2c+6a}+\frac{c(c^3+1)}{2a+6b}\geq\frac{1}{8}(a^3+b^3+c^3+3)$$

Problem 43A. Proposed by Alexandru Benescu

Let $ABCDA'B'C'D'$ be a cube , R the midpoint of BB', S the midpoint of AD and T on $C'D'$, such that $\frac{C'T}{D'T}=2$. Find $cos(\measuredangle SC, TR)$.

Problem 44A. Proposed by Gabriel Crisan

Let us consider the sum:

$$S_n=\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\cdots+\frac{n+2}{n!+(n+1)!+(n+2)!}.$$

Solve the equation:

$$4x=\left[\frac{11}{2}+S_{10}\right]-x^2,$$

where $[y]$ is the floor of $y\in\mathbb{R}$.

Solution Problem 44A

$$S_n=\sum_{k=1}^{n}\frac{k+2}{k!+(k+1)!+(k+2)!}=\sum_{k=1}^{n}\frac{k+2}{k![1+(k+1)+(k+1)(k+2)]}$$

$$S_n=\sum_{k=1}^{n}\frac{k+2}{k!(1+k+1+k^2+3k+2)}=\sum_{k=1}^{n}\frac{k+2}{k!(k^2+4k+4)}=\sum_{k=1}^{n}\frac{k+2}{k!(k+2)^2}=\sum_{k=1}^{n}\frac{1}{k!(k+2)}$$

$$S_n=\sum_{k=1}^{n}\frac{k+1}{(k+2)!}=\sum_{k=1}^{n}\frac{k+1+1-1}{(k+2)!}=\sum_{k=1}^{n}\frac{k+2-1}{(k+2)!}=\sum_{k=1}^{n}\left[\frac{k+2}{(k+2)!}-\frac{1}{(k+2)!}\right]$$

$$S_n = \sum_{k=1}^{n} \left[\frac{1}{(k+1)!} - \frac{1}{(k+2)!} \right]$$

$$S_{10} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{1}{11!} - \frac{1}{12!} = \frac{1}{2} - \frac{1}{12!}$$

Then the equation becomes:

$$4x = \left[\frac{11}{2} + \frac{1}{2} - \frac{1}{12!} \right] - x^2,$$

$$x^2 + 4x = \left[6 - \frac{1}{12!} \right]$$

$$x^2 + 4x - 5 = 0$$

with solutions $x_1 = 1$ and $x_2 = -5$.

Problem 45A. Proposed by Alexandru Benescu

Let a and b be positive real numbers such that $a(a + 2b - 2) = 8b(b + 1)$. Find the value of $\frac{a}{b+1}$.

Solution Problem 45A

$$a(a + 2b - 2) = 8b(b + 1) \Longrightarrow a^2 + 2ab - 2a = 8b^2 + 8b$$

and

$$a^2 + 2ab + b^2 = 9b^2 + 2a + 8b \Longrightarrow (a + b)^2 - (3b)^2 = 2(a + 4b)$$

and

$$(a + b - 3b)(a + b + 3b) = 2(a + 4b) \Longrightarrow (a - 2b)(a + 4b) = 2(a + 4b).$$

From the statement of the problem $a > 0$ and $b > 0 \Longrightarrow a + 4b > 0$. Thus, we can simplify with $a + 4b$. Finally:

$$a - 2b = 2 \Longrightarrow a = 2(b + 1) \Longrightarrow \frac{a}{b + 1} = 2.$$

Problem 46A. Proposed by Alexandru Benescu

Let a, b and c be positive real numbers such that $abc(a + b + c) \geq a^2 + b^2 + c^2$. Find the minimum of the expression $a^2 + b^2 + c^2$.

Solution Problem 46A

$$abc(a+b+c) \geq a^2+b^2+c^2 \Longrightarrow a^2bc+ab^2c+abc^2 \geq a^2+b^2+c^2$$

From AM-GM,

$$bc = \sqrt{b^2c^2} \leq \frac{b^2+c^2}{2}$$

Thus,

$$a^2 \cdot \frac{b^2+c^2}{2} + b^2 \cdot \frac{c^2+a^2}{2} + c^2 \cdot \frac{a^2+b^2}{2} \geq a^2+b^2+c^2 \Longrightarrow a^2b^2+b^2c^2+c^2a^2 \geq a^2+b^2+c^2.$$

Let us denote $x = a^2$, $y = b^2$, $z = c^2$. We this notation:

$$xy+yz+zx \geq x+y+z \Longrightarrow 3(xy+yz+zx) \geq 3(x+y+z).$$

By using the well-known inequality,

$$(x+y+z)^2 \geq 3(xy+yz+zx)$$

we obtain

$$(x+y+z)^2 \geq 3(x+y+z).$$

By using $x+y+z \neq 0$, the previous result becomes:

$$x+y+z \geq 3 \Longrightarrow a^2+b^2+c^2 \geq 3.$$

An example of a, b, c, for which the minimum sum of their squares is 3, is $a = b = c = 1$.

Problem 47A. Proposed by Max Jiang

Find all integer solutions $(a,b,c) \in \mathbb{Z}^3$ for

$$1 + \frac{3}{b} + \frac{2}{c} + \frac{6}{bc} = \frac{1}{a} + \frac{3}{ab} + \frac{2}{ac} + \frac{25}{abc}.$$

Solution Problem 47A

Note that any solution has $a, b, c \neq 0$. Thus, we have

$$\begin{aligned}
& 1 + \frac{3}{b} + \frac{2}{c} + \frac{6}{bc} = \frac{1}{a} + \frac{3}{ab} + \frac{2}{ac} + \frac{25}{abc} \\
\Leftrightarrow\ & abc + 3ac + 2ab + 6a = bc + 3c + 2b + 25 \\
\Leftrightarrow\ & a(bc+3c+2b+6) = (bc+3c+2b+6) + 19 \\
\Leftrightarrow\ & (a-1)(bc+3c+2b+6) = 19 \\
\Leftrightarrow\ & (a-1)(bc+2b+3c+6) = 19 \\
\Leftrightarrow\ & (a-1)(b(c+2)+3(c+2)) = 19 \\
\Leftrightarrow\ & (a-1)(b+3)(c+2) = 19.
\end{aligned}$$

Since $a, b, c \in \mathbb{Z}$ and 19 is prime, we must have $a-1, b+3, c+2$ be some permutation of $(1, 1, 19)$, $(-1, -1, 19)$, or $(-1, 1, -19)$. We can work through all 12 cases:

$(a-1, b+3, c+2)$	(a, b, c)
$(1, 1, 19)$	$(2, -2, 17)$
$(1, 19, 1)$	$(2, 16, -1)$
$(19, 1, 1)$	$(20, -2, -1)$
$(-1, -1, 19)$	$(0, -4, 17)$
$(-1, 19, -1)$	$(0, 16, -3)$
$(19, -1, -1)$	$(20, -4, -3)$
$(-1, 1, -19)$	$(0, -2, -21)$
$(-1, -19, 1)$	$(0, -22, -1)$
$(-19, -1, 1)$	$(-18, -4, -1)$
$(1, -1, -19)$	$(2, -4, -21)$
$(1, -19, -1)$	$(2, -22, -3)$
$(-19, 1, -1)$	$(-18, -2, -3)$

Problem 48A. Proposed by Vedaant Srivastava

Let G be the set of all lattice points (x, y) on the Cartesian plane where $0 \leq x, y \leq 2021$. Suppose that there are 2022 roadblocks positioned at points in G such that no two roadblocks have the same x or y coordinate. Anna starts at the point $(0, 0)$, attempting to reach the point $(2021, 2021)$ through a sequence of moves. In each move, she moves one unit up, down, left, or right, such that she always remains in G. Given that Anna cannot visit a lattice point which is occupied by a roadblock, determine all configurations of the roadblocks in which Anna is unable to reach her destination.

Problem 49A. Alexander Monteith-Pistor

Let T_1 and T_2 be triangles. We write $T_1 \hookrightarrow T_2$ if the interior of T_2 can be divided into triangles, all of which are similar to T_1. Prove that there exists an infinite set of triangles S such that, for any distinct $T, T' \in S$, $T \hookrightarrow T'$ is false.

Solution Problem 49A

Suppose $T_1 \hookrightarrow T_2$ and let α, β, γ be the angles of T_1. Then, by dividing the interior of T_2 into triangles similar to T_1, one also divides the angles of T_2 into angles of T_1. Therefore, if θ is an angle in T_2 then $\theta = a\alpha + b\beta + c\gamma$ for some non-negative integers a, b and c.

Now consider primes $p_1 < p_2 < ....$ For all $i \geq 1$, let T_i be a triangle with angles $\frac{1}{\sqrt{p_i}} \cdot 90°, \left(1 - \frac{1}{\sqrt{p_i}}\right) \cdot 90°$ and $90°$. We claim that $S = \{T_1, T_2, ...\}$ is a set satisying the desired conditions.

For the sake of contradiction, suppose $T_i \hookrightarrow T_j$ for some $i \neq j$. Then there are non-negative integers a, b, c such that

$$\frac{1}{p_j} \cdot 90^\circ = a \cdot \frac{1}{\sqrt{p_i}} \cdot 90^\circ + b \cdot \left(1 - \frac{1}{\sqrt{p_i}}\right) \cdot 90^\circ + c \cdot 90^\circ$$

simplifying,

$$\sqrt{p_i} = (a-b)\sqrt{p_j} + (b+c)\sqrt{p_i p_j}$$

Squaring both sides, the only irrational term has coefficient $2(a-b)(b+c)$. Therefore either $a = b$ or $b = c = 0$. In both cases, we quickly arrive at a contradiction (by obtaining that either $\sqrt{\frac{p_i}{p_j}}$ or $\sqrt{p_j}$ is irrational). In summary, the set $S = \{T_1, T_2, ...\}$ satisfies the desired conditions which completes the proof.

Problem 50A. Proposed by Nicholas Sullivan

Let O be the intersection point of the two diagonals of non-degenerate quadrilateral $ABCD$. Next, let E, F, G and H be the midpoints of AB, BC, CD and DA respectively. If EG and FH intersect at O, show that $ABCD$ is a parallelogram.

Problem 51A. Proposed by Andy Kim

Let a, b, c be positive reals. Prove

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c.$$

Solution Problem 51A

We begin by proving a lemma. Let x, y, w, z be positive reals, then we have

$$\begin{gathered} (xz - yw)^2 \geq 0 \\ x^2z^2 + y^2w^2 \geq 2xywz \\ x^2z(w+z) + y^2w(w+z) \geq (x^2 + 2xy + y^2)wz \\ \frac{x^2}{w} + \frac{y^2}{z} \geq \frac{(x+y)^2}{w+z} \end{gathered}$$

So, for all positive reals x, y, w, z, we have $\frac{x^2}{w} + \frac{y^2}{z} \geq \frac{(x+y)^2}{w+z}$.

Then, using this lemma twice, we have

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b)^2}{b+c} + \frac{c^2}{a} \geq \frac{(a+b+c)^2}{a+b+c} = a + b + c$$

as desired.

Problem 52A. Proposed by Andy Kim

Let n be a positive integer, and let

$$a_n = 1 \cdot \binom{n}{1} + \cdots + n \cdot \binom{n}{n} = \sum_{i=1}^{n} i \cdot \binom{n}{i}$$

$a)$ Prove that a_n is divisible by n.
$b)$ Find a value for a_n in terms of n.

Solution Problem 52A

Note that here we prove part $a)$ by proving part $b)$ first. One way to just prove $a)$ is to use the identity $\binom{n}{k} = \binom{n}{n-k}$ and match up the corresponding terms (handling the $k = n/2$ case for n even separately).

From the binomial formula, we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Then, taking the derivative of both sides, we have

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k \cdot \binom{n}{k} x^{k-1}$$

Finally, plugging in $x = 1$ gives

$$\sum_{k=1}^{n} k \cdot \binom{n}{k} = n2^{n-1}$$

(note that this is clearly divisible by n, proving part $a)$)

Problem 53A. Proposed by Nikola Milijevic

The positive integers $a_1, a_2, \ldots, a_n$ are not greater than 2021, with the property that $\text{lcm}(a_i, a_j) > 2021$ for all $i, j, i \neq j$. Show that:

$$\sum_{i=1}^{n} \frac{1}{a_i} < 2$$

Problem 39B. Proposed by Alexander Monteith-Pistor

For $n \in \mathbb{N}$, let $S(n)$ and $P(n)$ denote the sum and product of the digits of n (respectively). For how many $k \in \mathbb{N}$ do there exist positive integers $n_1, ..., n_k$ satisfying

$$\sum_{i=1}^{k} n_i = 2021$$

$$\sum_{i=1}^{k} S(n_i) = \sum_{i=1}^{k} P(n_i)$$

Problem 40B. Proposed by Vedaant Srivastava

Two identical rows of numbers are written on a chalkboard, each comprised of the natural numbers from 1 to 10! inclusive. Determine the number of ways to pick one number from each row such that the product of the two numbers is divisible by 10!

Problem 42B. Proposed by Andy Kim

Define an L-region of size n as an L-shaped region with two sides of length $2n$ and four sides of length n, and define an L-tile to be a tile with the same shape as an L-region of size 1 (i.e. a 2×2 square with one 1×1 square missing). Prove that an L-region of size n can be tiled with L-tiles for all positive integers n.

Problem 46B. Proposed by Ana Maria Popa

For positive numbers x, y, and z show that:

$$S = \frac{(x+z-y)^2}{4} + \frac{(x+y-z)^2}{4} + \frac{(y+z-x)^2}{4} + \frac{(x+z-y)^2}{4y^4} + \frac{(x+y-z)^2}{4z^4} + \frac{(y+z-x)^2}{4x^4} \geq \frac{3}{2}.$$

Solution Problem 46B.

We will use AM-GM: $\frac{a+b}{2} \geq \sqrt{ab}$

$$\frac{(x+z-y)^2}{4} + \frac{(x+z-y)^2}{4y^4} \geq 2\sqrt{\frac{(x+z-y)^2}{16y^4}} \implies \frac{(x+z-y)^2}{4} + \frac{(x+z-y)^2}{4y^4} \geq 2\frac{(x+z-y)^2}{4y^2}$$

We will also apply the inequality for the rest of the terms:

$$\frac{(x+y-z)^2}{4} + \frac{(x+y-z)^2}{4y^4} \geq 2\frac{(x+y-z)^2}{4y^2}$$

$$\frac{(y+z-x)^2}{4} + \frac{(y+z-x)^2}{4y^4} \geq 2\frac{(y+z-x)^2}{4y^2}$$

After using the inequality and replacing each term we obtain:

$$S \geq 2(\frac{(x+z-y)^2}{4y^2} + \frac{(x+y-z)^2}{4y^2} + \frac{(y+z-x)^2}{4y^2}).$$

We will use the Cauchy-Buniakovski-Schwartz Inequality:

$$(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$$

to obtain

$$(\frac{(x+z-y)^2}{4y^2}+\frac{(x+y-z)^2}{4y^2}+\frac{(y+z-x)^2}{4y^2})(1^2+1^2+1^2) \geq (\frac{x+z-y}{2y}+\frac{x+y-z}{2y}+\frac{y+z-x}{2y})^2$$

We will use now Nesbitt's inequality:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

to obtain

$$S \geq 2\frac{1}{3}(\frac{x+z-y}{2y}+\frac{x+y-z}{2y}+\frac{y+z-x}{2y})^2 \geq 2\frac{1}{3}(\frac{3}{2})^2 = \frac{3}{2}$$

Problem 47B. Proposed by Andrei Radu Vasile

Given a, b, c positive real numbers such that $a+b+c=15$, prove that:

$$\frac{a}{a^2+50}+\frac{b}{b^2+50}+\frac{c}{c^2+50} \leq \frac{1}{5}.$$

Solution Problem 47B

First, let us prove the following inequality: $x^2 \geq 10x-25$. It is equivalent to $x^2-10x+25 \geq 0$, which can be written as $(x-5)^2 \geq 0$, true of any real number x. Using this inequality, we can rewrite the terms in the following way:

$$\frac{x}{x^2+50} \geq \frac{x}{10x-25+50} = \frac{x}{10x+25}.$$

By replacing each term from the original problem, we get

$$\frac{a}{a^2+50}+\frac{b}{b^2+50}+\frac{c}{c^2+50} \geq \frac{a}{10a+25}+\frac{b}{10b+25}+\frac{c}{10c+25}.$$

Now, we prove that the second identity is less then $\frac{1}{5}$. We multiply by 10, then add 25 and subtract 25 from each fraction:

$$\frac{10a+25-25}{10a+25}+\frac{10b+25-25}{10b+25}+\frac{10c+25-25}{10c+25} \leq \frac{10}{5}$$

$$1-\frac{25}{10a+25}+1-\frac{25}{10b+25}+1-\frac{25}{10c+25} \leq 2$$

$$3-\frac{25}{10a+25}+\frac{25}{10b+25}+\frac{25}{10c+25} \leq 2$$

Subtract 2, then move the expression between parentheses on the right side:

$$\frac{25}{10a+25}+\frac{25}{10b+25}+\frac{25}{10c+25} \geq \frac{1}{25}$$

We rewrite the identity on the left such that we can use Titu Andreescu's inequality.

$$\frac{1^2}{10a+25}+\frac{1^2}{10b+25}+\frac{1^2}{10c+25} \geq \frac{(1+1+1)^2}{10(a+b+c)+75} = \frac{3^2}{10*15+75} = \frac{9}{225} = \frac{1}{25}$$

Problem 48B. Proposed by Alexandru Benescu

Let x and p be positive integers with p prime, such that:

$$x^{x^{x^{\cdots x}}} = p^{p^{p^{\cdots p}}},$$

where x appears p times and p appears x times. Prove that $x^2+p^2+2(x-p-xp)$ is nonnegative.

Solution Problem 48B

$$x^{x^{x^{\cdots x}}} = p^{p^{p^{\cdots p}}} \Longrightarrow x|p^{p^{p^{\cdots p}}}$$

with

$$x \in \left\{p^0, p^1, p^2, p^3, \cdots, p^{p^{p^{\cdots p}}}\right\}.$$

If

$$x = p^0 \Longrightarrow x = 1 \Longrightarrow p^{p^{p^{\cdots p}}} = 1 \Longrightarrow p = 1$$

(false, because p is prime). Then,

$$x \geq p^1 \Longrightarrow x^{x^{x^{\cdots x}}} \geq p^{p^{p^{\cdots p}}},$$

where p appears p times. Consequently,

$$p^{p^{p^{\cdots p}}} \geq p^{p^{p^{\cdots p}}},$$

where, on the left side, p appears x times and, on the right side, p appears p times $\Longrightarrow x \geq p$. However, $x^2+p^2+2(x-p-xp) = x^2+p^2+2x-2p-2xp = (x-p)^2+2(x-p) = (x-p)(x-p+2)$. From the last expression and from $x \geq p$ we can conclude $x^2+p^2+2(x-p-xp) \geq 0$. Consequently, $x^2+p^2+2(x-p-xp)$ is nonnegative.

Problem 49B. Proposed by Cosmina Ghitescu

Solve the equation $2021 + 2^x = 7^y 5^z$, where $x,y,z \in \mathbb{N}$.

Problem 50B. Proposed by Cosmina Ghitescu

Let a, b, c be the lengths of the sides of a triangle and a, b, $c \in \mathbb{Q}_+$, such that they verify the system:

$$\begin{cases} \frac{a}{b} + c = \frac{4(a+bc)}{(b+1)^2} \\ \frac{a+c}{b+1} + \sqrt{ac} = \frac{7+4\sqrt{3}}{10} \end{cases}$$

Find the area of the triangle.

Solution Problem 50B

We will use the following result: if $\forall x, y, z, t \in \mathbb{R}_+$ then $\frac{x}{z} + \frac{y}{t} \geq \frac{4(xt+yz)}{(z+t)^2}$ which is equivalent with $(z-t)^2(xt+zy) \geq 0$. If $z = t$ we have the equality. In our setting $x = a$, $y = c$, $z = b$, and $t = 1$. The condition $z = t$ corresponds to $b = 1$.
From the initial relationship $\frac{a+c}{b+1} + \sqrt{ac} = \frac{7+4\sqrt{3}}{10}$ we obtain:

$$\frac{a+c}{b+1} + \sqrt{ac} = \frac{7+4\sqrt{3}}{10} \Longrightarrow (\sqrt{a}+\sqrt{c})^2 = \frac{7+4\sqrt{3}}{10} = \left(\sqrt{\frac{3}{5}} + \sqrt{\frac{4}{5}}\right)^2$$

We know that $a,\, c \in \mathbb{Q}_+, \Longrightarrow a = \sqrt{\frac{3}{5}}, c = \sqrt{\frac{4}{5}}$. We notice that $a^2 + c^2 = b^2$. Thus, the triangle is right-angled with $area = \frac{ac}{2} = \frac{6}{25}$.

Problem 51B. Proposed by Cosmina Ghitescu

Find the minimum of the expression

$$E = \frac{1 - cos^2 A}{2cosA + cos^2 A + 1} + \frac{1 - cos^2 B}{2cosB + cos^2 B + 1} + \frac{1 - cos^2 C}{2cosC + cos^2 C + 1},$$

where A,B, and C are the angles of a triangle.

Solution Problem 51B

$$\frac{1 - cos^2 A}{2cosA + cos^2 A + 1} = \frac{sin^2 A}{2cosA + cos^2 A + 1} = \left(\frac{sinA}{1 + cosA}\right)^2$$

Thus, our expression becomes:

$$E = \frac{sinA}{1 + cosA} + \frac{sinB}{1 + cosB} + \frac{sinC}{1 + cosC}$$

Now, we will prove $\frac{sinA}{1+cosA} = tan\frac{A}{2}$.

$$sinA = sin(2 \cdot \frac{A}{2}) = \frac{2tan\frac{A}{2}}{1 + tan^2\frac{A}{2}}$$

$$1 + cosA = 1 + cos(2 \cdot \frac{A}{2}) = 1 + \frac{1 - tan^2\frac{A}{2}}{1 + tan^2\frac{A}{2}} = \frac{2}{1 + tan^2\frac{A}{2}}$$

We obtained the result: $\frac{sinA}{1+cosA} = tan\frac{A}{2}$. We can rewrite the initial expression:

$$E = \tan^2\frac{A}{2} + tan^2\frac{B}{2} + tan^2\frac{C}{2}$$

Using the inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$

with equality for $x = y = z$, we have:

$$E \geq \tan\frac{A}{2}tan\frac{B}{2} + \tan\frac{B}{2}tan\frac{C}{2} + \tan\frac{C}{2}tan\frac{A}{2} = 1$$

Consequently, the minimum of the expression is 1, for $tan\frac{A}{2} = tan\frac{B}{2} = tan\frac{C}{2}$.

Problem 52B. Proposed by Daisy Sheng

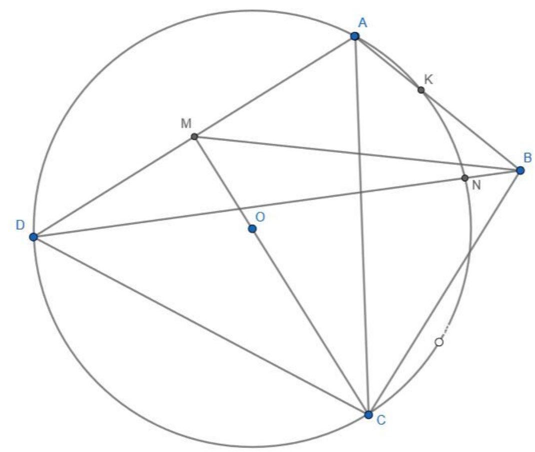
Find the general form for the integer k such that the expression

$$2^{n+1} + 5^{n+2} \cdot 3^{2n+4} \cdot k + (2k+1) \cdot (47 \cdot 3)^{n+3}$$

is divisible by 2021 for all positive integers n that are odd multiples of 3. For reference, $2021 = 43 \cdot 47$.

Problem 53B. Proposed by Daisy Sheng

Quadrilateral $ABCD$ is constructed with M as the midpoint of AD and $2AB > AD$. Let the circumcircle of triangle ACD intersect AB at K and BD at N, where arc $KN = 30°$ and $\measuredangle ABD = 45°$ (see figure below). If $CD^2 + 2AC \cdot AB = 4AB^2 + AC \cdot CD$ and $\cos(m(\measuredangle BAD)) = \frac{AB-AC}{AD}$, prove that $BM = CM$.
**Inspired by a TST Problem for Girls' Math Team Canada.*



Problem 54B. Proposed by Max Jiang

Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following property:

Given an ordered pair $(x, y) \in \mathbb{R}^2$, we have either

$$f(x) - f(y) = f(x^2 - y^2)$$
$$f(x)f(y) = f(x^2y^2)$$

or

$$\begin{aligned} f(y) &\neq 0 \\ f(x) + f(y) &= f(x^2 - y^2) \\ f(x)/f(y) &= f(x^2y^2). \end{aligned}$$

Note: it is possible that an ordered pair $(x, y), x \neq y$ satisfies the first set of conditions while the ordered pair (y, x) satisfies the other set.

Problem 55B. Proposed by Vedaant Srivastava

Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$2f(x + y) + 2f(x - y) = f(2x) + f(2y)$$

for all $x, y \in \mathbb{Q}$.

Solution Problem 55B

The answer is $f(x) \equiv ax^2$ for some $a \in \mathbb{Q}$. Let $P(x, y)$ denote the given assertion. From $P(0, 0)$, we obtain that

$$2f(0) + 2f(0) = f(0) + f(0) \iff 2f(0) = 0$$

So we have that $f(0) = 0$. Now, from $P(x/2, -x/2)$, we have that

$$2f(0) + 2f(x) = f(x) + f(-x) \iff f(x) = f(-x)$$

Furthermore, from $P(x, 0)$ we obtain

$$2f(x) + 2f(x) = f(2x) + f(0) \iff 4f(x) = f(2x)$$

This motivates the following claim:

Claim: For every positive integer k, $f(kx) = k^2f(x)$

Proof. We proceed by strong induction on k. We have already proved the result for the base cases of $k = 1$ and $k = 2$. Suppose the result holds for all positive integers up to k. Now, $P(kx, x)$ gives us

$$\begin{aligned} 2f((k + 1)x) + 2f((k - 1)x) &= f(2kx) + f(2x) \\ f((k + 1)x) &= (f(2kx) + f(2x) - 2(f(k - 1)x))/2 \\ f((k + 1)x) &= 2f(kx) + 2f(x) - (f(k - 1)x) \\ f((k + 1)x) &= 2k^2f(x) + 2f(x) - (k - 1)^2f(x) \quad \text{(IH)} \\ f((k + 1)x) &= (k + 1)^2f(x) \end{aligned}$$

so the induction step is complete. □

Let $f(1) = a$. Then we have that for positive integers p and q,

$$f\left(\frac{p}{q}\right) = p^2 \cdot f\left(\frac{1}{q}\right) = \frac{p^2}{q^2} \cdot f(1) = \frac{p^2}{q^2} \cdot a$$

So $f(x) = ax^2$ for all $x \in \mathbb{Q}^+$. As $f(-x) = f(x) = ax^2 = a(-x)^2$, then we have proved the result for all $x \in \mathbb{Q}$.

Problem 56B. Proposed by Alexander Monteith-Pistor

A game is played with white and black pieces and a chessboard (8 by 8). There is an unlimited number of identical black pieces and identical white pieces. To obtain a starting position, any number of black pieces are placed on one half of the board and any number of white pieces are placed on the other half (at most one piece per square). A piece is called matched if its color is the same of the square it is on. If a piece is not matched then it is mismatched. How many starting positions satisfy the following condition

$$\#\text{ of matched pieces} - \#\text{ of mismatched pieces} = 16$$

(your answer should be a binomial coefficient)

Problem 57B. Proposed by Nicholas Sullivan

Consider two perpendicular vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$. If these vectors have components $\mathbf{a} = (\sin\alpha, \sin\beta, \sin\gamma)$ and $\mathbf{b} = (\cos\alpha, \cos\beta, \cos\gamma)$ respectively, and

$$\sin^2(\alpha - \beta) + \sin^2(\beta - \gamma) + \sin^2(\gamma - \alpha) = 2,$$

then find $|\mathbf{a}|^4 + |\mathbf{b}|^4$.

Solution Problem 57B

Using the angle subtraction identity for sine, we know that:

$$\begin{aligned} \sin^2(\alpha - \beta) &= (\sin\alpha\cos\beta - \sin\beta\cos\alpha)^2 \\ &= \sin^2\alpha\cos^2\beta + \sin^2\beta\cos^2\alpha - 2\sin\alpha\cos\alpha\sin\beta\cos\beta. \end{aligned}$$

Thus, the entire expression can be rewritten:

$$\begin{aligned} &\sin^2(\alpha - \beta) + \sin^2(\beta - \gamma) + \sin^2(\gamma - \alpha) \\ &= \sin^2\alpha\cos^2\beta + \sin^2\beta\cos^2\alpha - 2\sin\alpha\cos\alpha\sin\beta\cos\beta \\ &+ \sin^2\beta\cos^2\gamma + \sin^2\gamma\cos^2\beta - 2\sin\beta\cos\beta\sin\gamma\cos\gamma \\ &+ \sin^2\gamma\cos^2\alpha + \sin^2\alpha\cos^2\gamma - 2\sin\gamma\cos\gamma\sin\alpha\cos\alpha. \end{aligned}$$

Next, by adding and subtracting

$$\sin^2\alpha\cos^2\alpha + \sin^2\beta\cos^2\beta + \sin^2\gamma\cos^2\gamma,$$

this can alternately be expressed as:

$$\begin{aligned}&\sin^2(\alpha-\beta)+\sin^2(\beta-\gamma)+\sin^2(\gamma-\alpha)\\&=(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)\\&+(\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma)^2.\end{aligned}$$

Now, since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a}\cdot\mathbf{b}=0$. If we let $A=|\mathbf{a}|^2$ and $B=|\mathbf{b}|^2$, then we know that:

$$\begin{aligned}&(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)\\&+(\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma)^2\\&=AB-0^2=2\end{aligned}$$

Similarly, by the Pythagorean identity, we know that

$$\begin{aligned}A+B&=(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)+(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)\\&=3.\end{aligned}$$

Since $|\mathbf{a}|^4+|\mathbf{b}|^4=A^2+B^2$, then the quantity we want is:

$$\begin{aligned}|\mathbf{a}|^4+|\mathbf{b}|^4&=(A+B)^2-2AB\\&=3^2-2(2)=5\end{aligned}$$