Nr 5 - March 2021 Problems

May 3, 2021

Problems

Problem 30A. Proposed by Alexander Monteith-Pistor

Let $A_1B_1, A_2B_2, A_3B_3, A_4B_4$ be four line segments of length 10. For each pair $1 \leq i < j \leq 4$, the line segments A_iB_i and A_jB_j intersect at point P_{ij}. Starting at A_1 and travelling along the four line segments, find the least upper bound for the distance one has to travel to pass through all 6 points of intersection (P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}).

Problem 33A. Proposed by DC

On circle with diameter AB, take two points C and D and build the intersection between AC and BD denoted as P. Prove that

$$BD^2 - BC^2 = AC \times PC + BD \times DP$$

Solution:

Build the perpendicular PE from P on AB with E on AB.
$\Delta APE \sim \Delta ABC$ (AA) $[m(\sphericalangle AEP) = m(\sphericalangle ACB) = 90^\circ$ and $\sphericalangle CAB$ is common]. We obtain $AC \times AP = AB \times AE$. Following the same rationale, $\Delta BPE \sim \Delta BAD$ (AA) $[m(\sphericalangle BEP) = m(\sphericalangle BDA) = 90^\circ$ and $\sphericalangle DBA$ is common]. We obtain $DB \times BP = AB \times EB$.
From the two above relationships, we obtain $AB^2 = AC \times AP + BD \times BP$.
We now replace the side AP with $AC - PC$ and side BP with $BD - DP$. The last relationship becomes:

$$AB^2 = AC(AC - PC) + BD(BD - DP)$$

Finally:

$$AB^2 = AC^2 - AC \times PC + BD^2 - BD \times DP$$

and

$$AC \times PC + BD \times DP = BD^2 - AB^2 + AC^2$$

$$AC \times PC + BD \times DP = BD^2 - BC^2$$

Problem 35A. Proposed by DC

In trapezoid ABCD, the bases are AB=7 cm and CD=3 cm. The circle with the origin at A and radius AD intersects diagonal AC at M and N. Calculate the value of the product $CM \times CN$.

Problem 36A. Proposed by Cosmina Ghitescu

Solve the equation

$$10x^2 + 7 = 18x + 11^y$$

where $x \in \mathbb{Z}$ and $y \in \mathbb{N}$.

Solution:

Let $\nu_p(n)$ denote the exponent of the prime p in the prime factorization of n.

Case 1 : $y = 0$

$$\begin{aligned} 10x^2 + 7 &= 18x + 1 \\ 10x^2 - 18x + 6 &= 0 \\ 5x^2 - 9x + 3 &= 0 \end{aligned}$$

By the quadratic formula, it can be verified that there are no solutions for $x \in \mathbb{Z}$.

Case 2 : $y = 1$

$$\begin{aligned} 10x^2 + 7 &= 18x + 11 \\ 5x^2 - 9x - 2 &= 0 \\ (x-2)(5x+1) &= 0 \end{aligned}$$

Thus the only solution where $x \in \mathbb{Z}$ is $x = 2$.

Case 3 : $y > 1$

$$\begin{aligned} 10x^2 + 7 &= 18x + 11^y \\ 10x^2 - 18x + (7 - 11^y) &= 0 \end{aligned}$$

The discriminant Δ can be computed for the above equation:

$$\begin{aligned} \Delta &= 324 - 4 \cdot 10(7 - 11^y) \\ &= 44 + 40 \cdot 11^y \\ &= 2^2 \cdot 11(1 + 10 \cdot 11^{y-1}) \end{aligned}$$

In order to have $x \in \mathbb{Z}$, we must have that Δ is a perfect square, implying that $\nu_{11}(\Delta)$ is even.

However,

$$\begin{aligned}\nu_{11}(\Delta) &= \nu_{11}(2^2 \cdot 11(1+10\cdot 11^{y-1})) \\ &= \nu_{11}(11) + \nu_{11}(1+10\cdot 11^{y-1}) \\ &= 1+0 \\ &= 1\end{aligned}$$

Contradiction. (The third line follows as $1+10\cdot 11^{y-1} \equiv 1 \not\equiv 0 \pmod{11}$)

Thus the only solution is $y=1$ and $x=2$.

Problem 37A. Proposed by Cosmina Ghitescu

Remark: The original submission was modified by AE.
Find all $x \in \mathbb{R}$ that satisfy the equation

$$\left[\frac{18x-4}{5}\right] + \left[\frac{36x-3}{10}\right] = \frac{2(p-1)!+5}{p}$$

where p is a prime number.

Solution:

Let $y = \frac{18x-4}{5}$.

We notice that $y + \frac{1}{2} = \frac{18x-4}{5} + \frac{1}{2} = \frac{36x-3}{10}$.

So we can rewrite the equation as

$$\begin{aligned}[y] + \left[y+\frac{1}{2}\right] &= \frac{2(p-1)!+5}{p} \\ [2y] &= \frac{2(p-1)!+5}{p} \quad \text{(Hermite)}\end{aligned}$$

From Wilson's Theorem we have $(p-1)! \equiv -1 \pmod{p}$

As $[2y] \in \mathbb{Z}$, we must have

$$\begin{aligned}2(p-1)!+5 &\equiv 0 \pmod{p} \\ 3 &\equiv 0 \pmod{p} \quad \text{(Wilson)} \\ p &= 3\end{aligned}$$

This yields

$$\begin{aligned}[2y] &= \frac{2(3-1)!+5}{3} \\ &= 3\end{aligned}$$

Finally, this implies that

$$3 \leq 2y < 4 \iff 3 \leq \frac{36x-8}{5} < 4 \iff \frac{23}{26} \leq x < \frac{7}{9}$$

$\therefore x \in \left[\frac{23}{36}, \frac{7}{9}\right)$

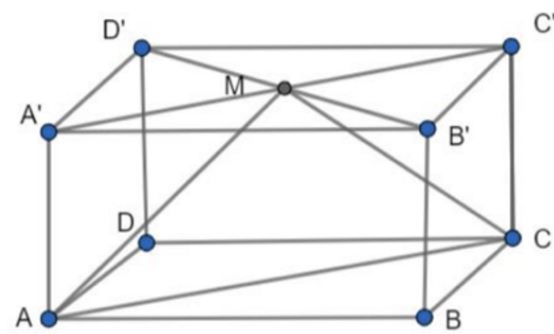
Problem 38A. Proposed by Cosmina Ghitescu

Remark: The original submission was modified by AE.

Let $ABCDA'B'C'D'$ be a square-based prism with square faces $ABCD$ and $A'B'C'D'$.
Let $AB = \sqrt{2}AA'$.
Consider a point M on the plane $(A'B'C'D')$ such that $\measuredangle AMC \geq 90^\circ$.

Prove that :

1. The measure of $\measuredangle AMC$ can only be 90°
2. M is the center of the base $A'B'C'D'$



Solution:

1. Let $AA' = a, AB = a\sqrt{2}$.
   As $\triangle ABC$ is isosceles right, we have $AC^2 = (AB\sqrt{2})^2 = 4a^2$.

   Using The Law of Cosines we have

   $$AC^2 = AM^2 + MC^2 - 2AM \cdot MC \cdot \cos \measuredangle AMC \tag{1}$$

   As $\triangle AA'M, CC'M$ are right:

   $$AM^2 = AA'^2 + A'M'^2 = a^2 + A'M^2$$

$$MC^2 = CC'^2 + C'M^2 = a^2 + C'M^2$$

Because $\measuredangle AMC \geq 90^\circ \Rightarrow -\cos\measuredangle AMC = |\cos\measuredangle AMC|$.

So we can rewrite (1) as

$$\begin{aligned} 4a^2 &= 2a^2 + A'M^2 + C'M^2 + 2AM \cdot MC \cdot |\cos\measuredangle AMC| \\ 2a^2 &= A'M^2 + C'M^2 + 2AM \cdot MC \cdot |\cos\measuredangle AMC| \end{aligned} \tag{2}$$

However, by the inequality $2(x^2 + y^2) \geq (x + y)^2$, we have

$$A'M^2 + MC'^2 \geq \frac{(A'M + MC')^2}{2} \geq \frac{A'C'^2}{2} = 2a^2 \tag{3}$$

Where $A'M + MC' \geq A'C'$ by the triangle inequality.

From (2) and (3):

$$\begin{aligned} 2a^2 &\geq 2a^2 + 2AM \cdot MC \cdot |\cos\measuredangle AMC| \\ 0 &\geq 2AM \cdot MC \cdot |\cos\measuredangle AMC| \\ 0 &= \cos\measuredangle AMC \qquad (AM, MC > 0) \\ &\boxed{90^\circ = \measuredangle AMC} \end{aligned}$$

2. From a) we have the equalities $A'M = MC'$ and $A'M + MC' = A'C' \Rightarrow A' - M - C'$ collinear.

   $\therefore M$ is the center of the base $A'B'C'D'$

Alternate solution, by the editors:

Let $AA' = r$.
Let O be the midpoint of AC.
Let M be the projection of O onto $(A'B'C'D')$. Observe that M is the centre of base $A'B'C'D'$.
Let S_1 be a sphere with centre O and diameter AC. Observe that the radius of S_1 is $OA = AC/2 = 2r/2 = r = OM$.

$\therefore$ As $OM = r \implies M \in S_1$, we have that $\measuredangle AMC = 90^\circ$.

By projection, for any point $M' \in (A'B'C'D') \mid M' \neq M$, we have that

$$OM' > OM = r$$

Thus all points M' lie outside S_1, implying that $\measuredangle AM'C < 90^\circ$.

$\therefore M$ is the only point satisfying $\measuredangle AMC \geq 90^\circ$.

Problem 39A. Proposed by Max Jiang

In a 2^n-player single-elimination tournament, the players are seeded from 1 to 2^n, where player i will always win against player j if $i < j$. In each round, the remaining players are paired up randomly. Find all pairs of players that are will be paired up at some point in the tournament no matter how the pairings are chosen each round.

Solution:

We claim the answer is that only the player 1 and player 2 match must occur.

Note that only player 1 can eliminate player 2 and that player 1 will always win the tournament since no one can eliminate him. Thus, player 2 must be eliminated from the tournament, meaning player 1 must be paired with player 2 at some point.

Now, for any other player i with $3 \leq i \leq 2^n$, note that player i could be paired with player 1 or player 2 in the first round. In this case, they will be eliminated in the first round and thus not be in any other pairs. Thus, it is possible that player i not be paired with player j for any given $j \neq i$, so no pairs involving a player $i \geq 3$ must occur.

Problem 40A. Proposed by Nicholas Sullivan

Let $a_0 = 1$, $a_1 = 1$ and $a_{n+1} = 2023a_n - a_{n-1}$, for all positive natural numbers n. Show that for all $n \geq 1$:

$$a_{n+1}a_{n-1} - a_n^2 = 2021.$$

Solution:

This can be proven by induction, where we let P_n be the proposition that $a_{n+1}a_{n-1} - a_n^2 = 2021$. Let us first consider the base case $n = 1$. Since $a_0 = 1$, $a_1 = 1$ and $a_2 = 2022$, then $a_2a_0 - a_1^2 = 2022 - 1^2 = 2021$. Thus, P_1 is true, and the base case is satisfied.

Now, for the induction step, we introduce the assumption that P_k is true, and show that P_{k+1} is also true. First, we take the left-hand side of the expression for $n = k + 1$, and simplify:

$$\begin{aligned} a_{k+2}a_k - a_{k+1}^2 &= (2021a_{k+1} - a_k)a_k - a_{k+1}^2 \\ &= a_{k+1}(2021a_k - a_{k+1}) - a_k^2 \\ &= a_{k+1}a_{k-1} + a_k^2. \end{aligned}$$

Since $a_{k+1}a_{k-1} - a_k^2 = 2021$ by the induction step, then:

$$a_{k+2}a_k - a_{k+1}^2 = 2021.$$

Thus, P_k implies P_{k+1} for all $k \in \mathbb{N}^+$. Since P_1 is true, then by induction, P_n is true for all $n \in \mathbb{N}^+$. Thus, for all $n \geq 1$, $a_{n+1}a_{n-1} - a_n^2 = 2021$.

Problem 41A. Proposed by Alexander Monteith-Pistor

Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 1$ and

$$f(p^k m) = \sum_{i=0}^{k-1} f(p^i m)$$

for all $p, k, m \in \mathbb{N}$ where p is a prime which does not divide m.

Problem 42A. Proposed by Vedaant Srivastava

Given positive reals a, b, c, prove that

$$\frac{a(a^3+1)}{2b+6c} + \frac{b(b^3+1)}{2c+6a} + \frac{c(c^3+1)}{2a+6b} \geq \frac{1}{8}(a^3+b^3+c^3+3)$$

Problem 37B. Proposed by Max Jiang

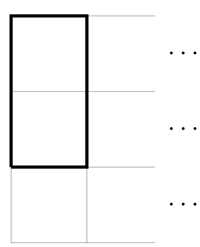
For an even integer n, find the number of ways to tile a $3 \times n$ grid with dominoes in terms of n.

Solution:

Let A_k be the number of ways to tile a $3 \times 2k$ grid. Then, let B_k be the number of ways to tile a $3 \times (2k-1)$ grid with a square attached to the left of the top- or bottom-left square.

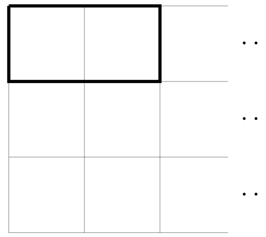
Let us consider the ways we can tile the top-left square in a $3 \times 2k$ grid.

Case 1: Tile with a vertical domino.



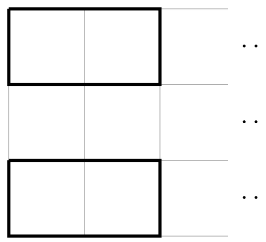
We see that we are left with B_k ways to tile the rest of the grid.

Case 2: Tile with a horizontal domino.

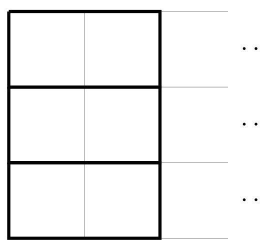


Then, there are two ways to tile the bottom right square.

Case 2.1: Tile with a horizontal domino.

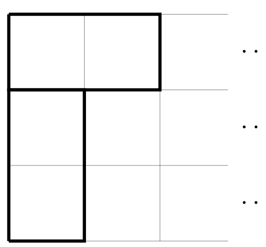


Then, we must tile the middle leftmost square with another horizontal domino.

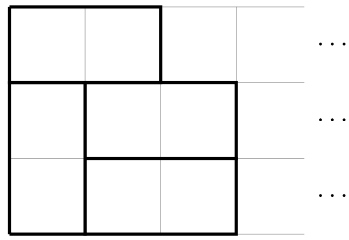


Now, there are A_{k-1} ways to tile the rest of the grid.

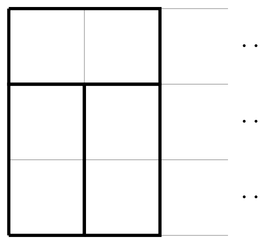
Case 2.2: Tile with a vertical domino.



Then, there are two ways to tile the remaining two squares in the second-rightmost column. Like



Leaving us with B_{k-1} ways to tile the rest of the grid, or

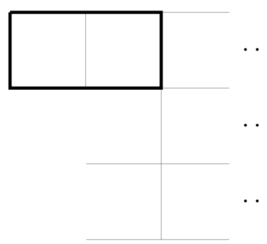


Leaving us with A_{k-1} ways to tile the rest of the grid.

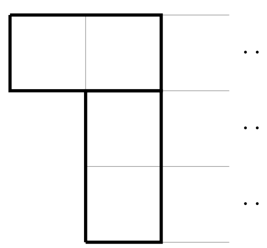
Combining these cases, we see that overall, we have

$$A_k = B_k + A_{k-1} + B_{k-1} + A_{k-1}.$$

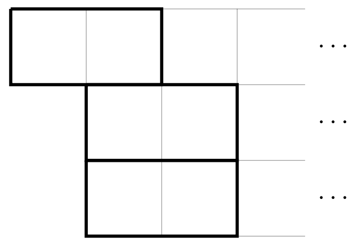
To find a recurrence relation for B_k, first note that there is only one way to tile the protruding square:



Then, we can tile the remaining two squares in the rightmost column like



Leaving us with A_{k-1} ways to tile the rest of the grid, or



Leaving us with B_{k-1} ways to tile the rest of the grid.

Combining these cases, we have

$$B_k = A_{k-1} + B_{k-1}.$$

Thus, our series satisfies

$$A_k = B_k + A_{k-1} + B_{k-1} + A_{k-1} \quad (4)$$
$$B_k = A_{k-1} + B_{k-1} \quad (5)$$

(2) gives $A_{k-1} = B_k - B_{k-1}$. Substituting this into (1) gives

$$(B_{k+1} - B_k) = B_k + (B_k - B_{k-1}) + B_{k-1} + (B_k - k_{n-1})$$
$$\implies B_{k+1} - 4B_k + B_{k-1} = 0.$$

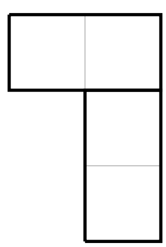
The solution to this linear homogeneous recurrence is

$$B_k = c_1 r_1^k + c_2 r_2^k$$

where c_1, c_2 are real constants and r_1, r_2 are the roots of the quadratic

$$x^2 - 4x + 1 = 0.$$

By the quadratic formula, these roots are $2 \pm \sqrt{3}$. To solve for c_1, c_2, consider when $k = 1$, giving us the unique tiling



and when $k = 2$, where we can confirm there are 4 tilings. Thus, we must have

$$c_1(2+\sqrt{3}) + c_2(2-\sqrt{3}) = 1$$
$$c_1(2+\sqrt{3})^2 + c_2(2-\sqrt{3})^2 = 4.$$

Solving the system yields $(c_1, c_2) = (1/2\sqrt{3}, -1/2\sqrt{3})$, so

$$B_k = \frac{1}{2\sqrt{3}}(2+\sqrt{3})^k - \frac{1}{2\sqrt{3}}(2-\sqrt{3})^k.$$

Then, we have

$$\begin{aligned}
A_k &= B_{k+1} - B_k \\
&= \left(\frac{1}{2\sqrt{3}}(2+\sqrt{3})^{k+1} - \frac{1}{2\sqrt{3}}(2-\sqrt{3})^{k+1}\right) - \left(\frac{1}{2\sqrt{3}}(2+\sqrt{3})^k - \frac{1}{2\sqrt{3}}(2-\sqrt{3})^k\right) \\
&= \frac{3+\sqrt{3}}{6}(2+\sqrt{3})^k + \frac{3-\sqrt{3}}{6}(2-\sqrt{3})^k.
\end{aligned}$$

Thus, the number of ways to tile a $3 \times n$ grid, where n is even, is

$$\boxed{\frac{3+\sqrt{3}}{6}(2+\sqrt{3})^k + \frac{3-\sqrt{3}}{6}(2-\sqrt{3})^k}$$

where $k = n/2$.

Problem 38B. Proposed by Nicholas Sullivan

Alice and Bob are playing a game called 'knights of the toroidal table' on a five-by-five square 'chessboard'. Each has a knight, which begin in opposite corners of the board. On each turn, the knight can move as a regular knight, that is, in an L-shape of two steps in one direction, and one step perpendicularly. However, if a knight goes over an edge, it reenters on the opposite side, as if the board were a torus.

Players make turns, and as soon as one player's knight is captured, or enters a square previously occupied by another knight, then this player loses the game, and the other player wins. If Alice moves first, who has the winning strategy, and what is it?

Solution:

Let us denote squares by counting from the upper-left corner, so that (a, b) refers to the square in the ath column and bth row, starting from $(0, 0)$ to $(4, 4)$.

If Alice moves first, then Bob has the winning strategy. Since Alice and Bob's knights begin in opposite corners of a 5x5 board, we can assume that Alice's knight is in the bottom left corner $(0, 4)$, and Bob's knight is in the top right corner $(4, 0)$. Bob's winning strategy is to mirror Alice's previous move, reflected across the upper-left to lower-right diagonal. In other words, if Alice plays (a, b), Bob should play (b, a).

To verify that this is a winning strategy, we need to first verify that it is always possible for Bob to make this move without losing. First, we recognize that if Alice plays on a square (a, a), on the diagonal, then Bob can also play at (a, a), taking Alice's knight and winning the game. Next, we see that for any square (a, b), it has been previously occupied if and only if (b, a) has been previously occupied, provided that Bob has been playing this strategy. Thus, if Alice can play at (a, b) for some $a \neq b$ without losing, then Bob can also play at (b, a) without losing.

Finally, we must ensure that Alice cannot take Bob's knight when Alice is at (a, b) and Bob is at (b, a). A knight's move from (a, b) would take Alice's knight to any square $(a \pm 1, b \pm 2) \mod 5$ or $(a \pm 2, b \pm 1) \mod 5$. Thus, if Alice's knight can take Bob's knight, then there are two possible cases. In the first case, $b = a \pm 2 \mod 5$ and $a = b \pm 1 \mod 5$, so:

$$b - a = \pm 2 = -(\pm 1) \mod 5.$$

Since $\pm 2 \neq \pm 1 \mod 5$, then this cannot be true. In the second case, $b = a \pm 1 \mod 5$ and $a = b \pm 2 \mod 5$, so:

$$b - a = \pm 1 = -(\pm 2) \mod 5.$$

Again, since $\pm 2 \neq \pm 1 \mod 5$, this cannot be true. Thus, Alice cannot take Bob's piece in this position.

Since Bob's move is never losing, then Alice will eventually run out of previously unoccupied spaces on the board, and will be forced to either move onto the diagonal and be taken, or a previously occupied square and lose. Thus, Bob has the winning strategy.

Problem 39B. Proposed by Alexander Monteith-Pistor

For $n \in \mathbb{N}$, let $S(n)$ and $P(n)$ denote the sum and product of the digits of n (respectively). For how many $k \in \mathbb{N}$ do there exist positive integers $n_1, ..., n_k$ satisfying

$$\sum_{i=1}^{k} n_i = 2021$$

$$\sum_{i=1}^{k} S(n_i) = \sum_{i=1}^{k} P(n_i)$$

Problem 40B. Proposed by Vedaant Srivastava

Two identical rows of numbers are written on a chalkboard, each comprised of the natural numbers from 1 to 10! inclusive. Determine the number of ways to pick one number from each row such that the product of the two numbers is divisible by 10!

Problem 41B. Proposed by Nikola Milijevic

The positive integers $a_1, a_2, \ldots, a_n$ are not greater than 2021, with the property that $\text{lcm}(a_i, a_j) > 2021$ for all $i, j, i \neq j$. Show that:

$$\sum_{i=1}^{n} \frac{1}{a_i} < 2$$

Solution:

We let k_1 be the number of a_i in the interval $(\frac{2021}{2}, 2021]$, k_2 the number of a_i in the interval $(\frac{2021}{3}, \frac{2021}{2}]$, and so on. Note if we have $\frac{2021}{m+1} < a_i \leq \frac{2021}{m}$ for some m and i, then $a_i, 2a_i, \ldots, ma_i$ are all no greater than 2021. Since $\text{lcm}(a_i, a_j) > 2021$, $k_1 + 2k_2 + 3k_3 + \ldots$ is the number of distinct integers no greater than 2021, that are multiples of one of the a_i. We have:

$$\begin{aligned} 2k_1 + 3k_2 + 4k_3 + \ldots &= (k_1 + k_2 + k_3 + \ldots) + (k_1 + 2k_2 + 3k_3 + \ldots) \\ &\leq n + 2021 \\ &\leq 4042 \end{aligned}$$

Finding an upper bound for the summation,

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{a_i} &< k_1 \frac{2}{2021} + k_2 \frac{3}{2021} + k_3 \frac{4}{2021} + \ldots \\ &= \frac{2k_1 + 3k_2 + 4k_3 + \ldots}{2021} \\ &\leq \frac{4042}{2021} \\ &= 2 \end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} \frac{1}{a_i} < 2$$

Problem 42B. Proposed by Andy Kim

Define an L-region of size n as an L-shaped region with two sides of length $2n$ and four sides of length n, and define an L-tile to be a tile with the same shape as an L-region of size 1 (i.e. a 2×2 square with one 1×1 square missing). Prove that an L-region of size n can be tiled with L-tiles for all positive integers n.

Problem 43B. Proposed by Andy Kim

For $n \in \mathbb{Z}^+ \cup \{0\}$, let $[\![n]\!] = \{i \in \mathbb{Z} \,|\, 1 \leq i \leq n\}$. Also, for a set of positive integers A, let $S(A)$ be the sum of the elements of A. Find (and prove) a formula for

$$\sum_{A \subseteq [\![n]\!]} \sum_{B \subseteq [\![n]\!]} S(A \cup B)$$

Solution:

We claim that

$$\sum_{A\subseteq[\![n]\!]}\sum_{B\subseteq[\![n]\!]} S(A\cup B) = 3\binom{n+1}{2}4^{n-1}$$

We proceed by induction.

Base case: $n = 0$

Since $[\![0]\!] = \varnothing$, we have

$$\sum_{A\subseteq[\![0]\!]}\sum_{B\subseteq[\![0]\!]} S(A\cup B) = S(\varnothing\cup\varnothing) = 0 = 3\cdot\binom{1}{2}\cdot 4^{0-1}$$

Inductive step:

Suppose the claim is true for some $k \in \mathbb{Z}^+ \cup \{0\}$.

Note that every subset of $[\![k+1]\!]$ either contains $k+1$ or it does not. Furthermore, the subsets do contain $k+1$ are in a one-to-one relationship with subsets of $[\![k]\!]$, formed by removing/adding $k+1$. So, we have

$$\begin{aligned}
&\sum_{A\subseteq[\![k+1]\!]}\sum_{B\subseteq[\![k+1]\!]} S(A\cup B)\\
&= \sum_{\substack{A\subseteq[\![k+1]\!],\\ k+1\notin A}}\sum_{\substack{B\subseteq[\![k+1]\!],\\ k+1\notin A}} S(A\cup B) + \sum_{\substack{A\subseteq[\![k+1]\!],\\ k+1\in A}}\sum_{\substack{B\subseteq[\![k+1]\!],\\ k+1\notin B}} S(A\cup B)\\
&\quad + \sum_{\substack{A\subseteq[\![k+1]\!],\\ k+1\notin A}}\sum_{\substack{B\subseteq[\![k+1]\!],\\ k+1\in B}} S(A\cup B) + \sum_{\substack{A\subseteq[\![k+1]\!],\\ k+1\in A}}\sum_{\substack{B\subseteq[\![k+1]\!],\\ k+1\in B}} S(A\cup B)\\
&= \sum_{A\subseteq[\![k]\!]}\sum_{B\subseteq[\![k]\!]} S(A\cup B) + \sum_{A'\subseteq[\![k]\!]}\sum_{B\subseteq[\![k]\!]} S((A'\cup\{k+1\})\cup B)\\
&\quad + \sum_{A\subseteq[\![k]\!]}\sum_{B'\subseteq[\![k]\!]} S(A\cup(B'\cup\{k+1\})) + \sum_{A'\subseteq[\![k]\!]}\sum_{B'\subseteq[\![k]\!]} S((A'\cup\{k+1\})\cup(B'\cup\{k+1\}))\\
&= 4\sum_{A\subseteq[\![k]\!]}\sum_{B\subseteq[\![k]\!]} S(A\cup B) + 3\sum_{A\subseteq[\![k]\!]}\sum_{B\subseteq[\![k]\!]}(k+1)\\
&= 4\cdot 3\binom{k+1}{2}4^{k-1} + 3(k+1)4^k\\
&= 3\binom{k+1}{2}4^k + 3\binom{k+1}{1}4^k\\
&= 3\binom{k+2}{2}4^k
\end{aligned}$$

and so the claim holds for $k+1$.

By induction, the claim is true for all $n \in \mathbb{Z}^+ \cup \{0\}$.

Problem 44B. Proposed by DC

Consider triangle ABC with $\measuredangle ABC = 30°$ and $\measuredangle ACB = 15°$ and M the midpoint of the side BC. Build AN, the angle bisector of $\measuredangle MAC$, with N on BC. Calculate the ratio $\frac{NC}{AB}$.

Solution:

Build PM the perpendicular bisector with P on AB. ΔBPC is isosceles with $n(\measuredangle PCB) = 30°$.

AC is angle bisector; consequently $\frac{AP}{AB} = \frac{CP}{CB}$. We have $CP = PB$ and $\frac{AP}{AB} = \frac{PB}{CB}$. We obtain $AB = \frac{AP \cdot BC}{PB}$.

From $\frac{AP}{AB} = \frac{CP}{CB}$ we can replace the terms CP with $2PM$ (from the 30-60-90 ΔMPC) and BC with $2BM$. We obtain $\frac{AP}{AB} = \frac{PM}{BM}$ and AM angle bisector in ΔBMP. If $\measuredangle BMA = 45°$ then $\measuredangle MAC = 30°$ ($\measuredangle BMA = \measuredangle MAC + \measuredangle ACM$). If AN, the angle bisector of $\measuredangle MAC$, then $\measuredangle NAC = 15°$ and $AN \parallel PC$; consequently $\frac{NC}{BC} = \frac{AP}{PB}$. We obtain $NC = \frac{AP \cdot BC}{PB}$.

Finally, $\frac{NC}{AB} = 1$

Problem 45B. Proposed by DC

In triangle ABC with $\measuredangle B = 30°$ prove that $sin(A) + cos(C) \leq \sqrt{3}$.

Solution:

$sin(A)+cos(C) = sin(A)+cos(90°-C) = 2sin\frac{A+90°-C}{2}cos\frac{A-90°+C}{2} = 2sin(45°+\frac{A-C}{2}cos\frac{180°-B-90°}{2} = 2sin(45° + \frac{A-C}{2})sin\frac{B}{2} = \sqrt{3}sin(45° + \frac{A-C}{2}) \leq \sqrt{3}$.