Number 3 - December 2020 Problems

January 2, 2021

Problem 15A. Proposed by Alexander Monteith-Pistor

Let $x_0 = 0$. For $n \geq 1$: if x_{n-1} is even, x_n is randomly chosen from the multiples of 3 between $x_{n-1} - 10$ and $x_{n-1} + 10$ (inclusive); if x_{n-1} is odd, x_n is randomly chosen from the multiples of 4 between $x_{n-1} - 10$ and $x_{n-1} + 10$ (inclusive). Find the probability that x_5 is even.

Solution

We first make a few observations:

- if $x_k = i$, $0 \leq k$ then i is either a multiple of 3 or a multiple of 4
- let $0 \leq k < n$. The probability that x_n is even given $x_k = i$ is equal to the probability that x_n is even given $x_k = -i$
- let $0 \leq k < n$. Translating x_k by 12 has no effect on the problem. Thus, the probability that x_n is even given $x_k = i$ is equal to the probability that x_n is even given $x_k = i + 12$

These properties follow directly from the problem statement. Notably, the probability $x_{k+1} = i_2$ given $x_k = i_1$ is equal to the probability $x_{k+1} = -i_2$ given $x_k = -i_1$. Similarly the probability $x_{k+1} = i_2$ given $x_k = i_1$ is equal to the probability $x_{k+1} = i_2 + 6$ given $x_k = i_1 + 6$.

Let a_n denote the probability that x_n is an even multiple of 3. Let b_n denote the probability that x_n is an odd multiple of 3. Finally, let c_n denote the probability that x_n is a multiple of 4 but not a multiple of 3. From the construction of x_n's in the problem statement, $a_0 = 1$ and $b_0 = c_0 = 0$. Further, for $n \geq 1$,

$$a_n = \frac{3}{7}a_{n-1} + \frac{2}{5}b_{n-1} + \frac{4}{7}c_{n-1}$$

$$b_n = \frac{4}{7}a_{n-1} + \frac{3}{7}c_{n-1}$$

$$c_n = \frac{3}{5} b_{n-1}$$

The above relationships hold because of the previous observations. Further, note that we can use the last relationship to obtain recurrences involving only the sequences (a_n) and (b_n):

$$a_n = \frac{3}{7} a_{n-1} + \frac{2}{5} b_{n-1} + \frac{12}{35} b_{n-2}$$

$$b_n = \frac{4}{7} a_{n-1} + \frac{9}{35} b_{n-2}$$

Note $a_n + b_n + \frac{3}{5} b_{n-1} = a_n + b_n + c_n = 1$ for all $n \geq 0$ (by the first observation). Using this, we obtain

$$b_n = \frac{4}{7}\left(1 - b_{n-1} - \frac{3}{5} b_{n-2}\right) + \frac{9}{35} b_{n-2} = \frac{4}{7} - \frac{4}{7} b_{n-1} - \frac{3}{35} b_{n-2}$$

Note we are now in a position to compute b_5 directly:

n	0	1	2	3	4	5
b_n	0	$\frac{4}{7}$	$\frac{12}{7^2}$	$\frac{656}{5 \cdot 7^3}$	$\frac{3984}{5 \cdot 7^4}$	$\frac{146644}{5^2 \cdot 7^5}$

Therefore the probability that x_5 is even is

$$1 - b_5 = 1 - \frac{146644}{5^2 \cdot 7^5} = \boxed{\frac{273531}{5^2 \cdot 7^5}} = \frac{273531}{420175} \approx 0.651$$

Note that it is possible to find a closed formula for b_n and thus for a_n, c_n and the probability that x_n is even. One could do this using generating functions.

Problem 20A. Proposed by DC

In a cyclic quadrilateral ABCD, P is the intersection of the diagonals and E, M, G and L are the midpoints on sides AB, BC, CD and DA. By projecting the medians PE, PM, PG and PL on sides AB, BC, CD and DA, four segments are obtained: EF, MN, GH and LI. Prove that

$$AB \times FE + CD \times HG = AD \times IL + CB \times MN.$$

Solution

In the cyclic quadrilateral ABCD, using the power of the point P: $AP \times CP = BP \times DP$.
We can derive $AP \times (AC - AP) = BP \times (BD - BP)$
$AP \times AC - AP^2 = BP \times BD - BP^2$
$AP \times AC - BP \times BD = AP^2 - BP^2$ (1)
Starting again from $AP \times CP = BP \times DP$.
$(AC - CP) \times CP = (BD - DP) \times DP$

$AC \times CP - CP^2 = (BD \times DP - DP^2)$
$AC \times CP - BD \times DP = CP^2 - DP^2$ (2)
By adding relationship (1) and (2) we obtain:
$AC \times (AP + CP) - BD \times (BP + DP) = AP^2 - BP^2 + CP^2 - DP^2$
$AC^2 - BD^2 = AP^2 - BP^2 + CP^2 - DP^2$ (3).
In $\Delta APB: AP^2 - BP^2 = 2AB \times FE$
obtained by applying Pythagorean theorem in ΔAPF and ΔAPF and using also that E is the midpoint of AB.
In $\Delta CPD: CP^2 - DP^2 = 2CD \times HG$
Consequently, (3) becomes: $AC^2 - BD^2 = 2(AB \times FE + CD \times HG)$ (4).
Applying again in ΔBPC and ΔAPD (3) becomes: $AC^2 - BD^2 = 2(BC \times MN + AD \times IL)$ (5).
From (4) and (5):

$$AB \times FE + CD \times HG = AD \times IL + CB \times MN.$$

Problem 21A. Proposed by Max Jiang

Alice, Bob, Carl, and Daniel like flipping coins. They are playing a game where they repeatedly flip coins simultaneously until each player has gotten at least 1 heads. What is the expected value of the number of times each player will flip his or her coin?

Solution

Let u_k represent the expected number of flips left if k players have not flipped heads yet. Note that $u_0 = 0$. Then, we have

$$u_1 = 1 + \frac{1}{2}u_1 + \frac{1}{2}u_0 = 1 + \frac{1}{2}u_1,$$

since after 1 flip, there is a $\frac{1}{2}$ chance the remaining player still has not flipped heads and a $\frac{1}{2}$ chance that we are done. Solving this equation yields $u_1 = 2$. Next, we have

$$u_2 = 1 + \frac{1}{4}u_2 + \frac{1}{2}u_1 + \frac{1}{4}u_0,$$

since there is a $\frac{1}{4}$ chance neither of the 2 remaining players flip heads, a $\frac{1}{4}$ chance both flip heads, and a $\frac{1}{2}$ that one of them does. Substituting $u_1 = 2$ and solving yields $u_2 = \frac{8}{3}$.

Similarly, we have

$$\begin{aligned}
u_3 &= 1 + \frac{1}{8}u_3 + \frac{3}{8}u_2 + \frac{3}{8}u_1 \\
\Rightarrow \frac{7}{8}u_3 &= 1 + 1 + \frac{3}{4} \\
\Rightarrow u_3 &= \frac{22}{7}, \\
u_4 &= 1 + \frac{1}{16}u_4 + \frac{1}{4}u_3 + \frac{3}{8}u_2 + \frac{1}{4}u_1 \\
\Rightarrow \frac{15}{16}u_4 &= 1 + \frac{11}{14} + 1 + \frac{1}{2} \\
\Rightarrow u_4 &= \frac{368}{105}.
\end{aligned}$$

So our final answer is $\boxed{\frac{368}{105}}$.

Problem 22A. Proposed by Nikola Milijevic

For a fixed natural number n, which k would maximize the following expression: $\binom{3n+k}{2n}\binom{3n-k}{2n}$

Solution

This expression equates to

$$\frac{(3n+k)!(3n-k)!}{(2n)!^2(n+k)!(n-k)!}$$

We can further write this as

$$\frac{1}{(2n)!^2}\Big((3n+k)(3n+k-1)\cdots(n+k+1)\Big)\Big((3n-k)(3n-k+1)\cdots(n-k+1)\Big)$$

$$\frac{1}{(2n)!^2}\Big((3n+k)(3n-k)\Big)\Big((3n+1+k)(3n+1-k)\Big)\ \cdots\Big((n+1+k)(n+1-k)\Big)$$

By the AM-GM inequality, this expresion is maximized for $k=0$, yielding a value of $\binom{3n}{2n}^2$

Problem 23A. Proposed by Nikola Milijevic

For how many natural numbers n is the expression $n!+3$ a perfect cube?

Solution

With a quick check of the numbers $1-8$, we see only when $n=4$ will $n!+3$ be a perfect cube

n	1	2	3	**4**	5	6	7	8
$n!+3$	4	5	9	**27**	123	723	5043	40323

We now show no such n exist for $n>8$. For $n>8$, $n!$ is clearly a multiple of three, so the sum of $n!$ and 3 must also be a multiple of three. For the sum to be a perfect cube, then $n!+3$ must be a multiple of 27. However, $n!$ is a multiple of 27 for $n>8$ but three is not. Therefore the sum cannot be a multiple 27 and cannot be a perfect square.

Therefore the ony such number is $n=4$

Problem 24A. Proposed by Vedaant Srivastava

Let $\{a_1, a_2, a_3, \ldots\}$ be sequence of rational numbers such that $a_1 = 2$ and

$$a_n = \frac{3}{2}\left(\frac{a_{n-1}}{3} + \frac{1}{a_{n-1}}\right)$$

for $n \geq 2$. Determine an explicit formula (in terms of n) for a_n.

Solution

From the recurrence relation, we have that

$$a_n = \frac{a_{n-1}^2 + 3}{2a_{n-1}}$$

Therefore we obtain that

$$\frac{a_n + \sqrt{3}}{a_n - \sqrt{3}} = \frac{\frac{a_{n-1}^2 + 2\sqrt{3}a_{n-1} + 3}{2a_{n-1}}}{\frac{a_{n-1}^2 - 2\sqrt{3}a_{n-1} + 3}{2a_{n-1}}} = \frac{(a_{n-1} + \sqrt{3})^2}{(a_{n-1} - \sqrt{3})^2}$$

By unfolding this new recurrence relation, we obtain that

$$\frac{a_n + \sqrt{3}}{a_n - \sqrt{3}} = \left(\frac{a_1 + \sqrt{3}}{a_1 - \sqrt{3}}\right)^{2^{n-1}} = (2+\sqrt{3})^{2^n}$$

Rearranging and solving for a_n, we get that

$$\boxed{a_n = \frac{\sqrt{3}\left((2+\sqrt{3})^{2^n} + 1\right)}{(2+\sqrt{3})^{2^n} - 1}}$$

Remark: An observant student would have noticed that the numerator and denominator of the successive fractional representations of a_n satisfy the Pell equation $x^2 - 3y^2 = 1$. It is left as an exercise to determine the explicit form of a_n by investigating the solution pairs corresponding to θ^{2^k}, where $\theta = 2 + \sqrt{3}$ is the generator of the equation.

Problem 25A. Proposed by Nicholas Sullivan

Consider isosceles triangle ABC ($AB = AC$, with point D on AB and E on AC such that $AD = DE = EB = BC$. Find $\mathrm{m}(\angle ABC)$.

Solution

Let $\alpha = \mathrm{m}(\angle ABC) = \mathrm{m}(\angle ACB)$. Since triangle BCE is also isosceles ($BE = BC$), then $\mathrm{m}(\angle BEC) = \alpha$, and $\mathrm{m}(\angle CBE) = \pi - 2\alpha$. Thus, $\mathrm{m}(\angle DBE) = \mathrm{m}(\angle ABC) - \mathrm{m}(\angle CBE) = 3\alpha - \pi$. Since triangle EBD is also isosceles ($EB = ED$), then $\mathrm{m}(\angle BDE) = 3\alpha - \pi$. Thus, $\mathrm{m}(\angle ADE) = \pi - \mathrm{m}(\angle BDE) = 2\pi - 3\alpha$. Since triangle DAE is also an isosceles triangle, then $\mathrm{m}(\angle DAE) = \frac{1}{2}(\pi - \mathrm{m}(\angle ADE)) = \frac{3}{2}\alpha - \frac{\pi}{2}$. Finally, since triangle ABC is isosceles, $\mathrm{m}(\angle DAE) = (\pi - 2\mathrm{m}(\angle ABC)) = \pi - 2\alpha$. Thus, $\pi - 2\alpha = \frac{3}{2}\alpha - \frac{\pi}{2}$, so $7\alpha = 3\pi$, and $\alpha = \frac{3}{7}\pi$.

Problem 26A. Proposed by Frederick Pu

Remark: The original submission was modified by AE.
Suppose you lived on an island where every islander x can be described by an ordered list of 169 real number attributes, $(x_1, x_2, \ldots, x_{169})$. One of the islanders, Bob, has attributes $B = (1, 2, \ldots, 169)$. We define an intelligence function, $I : \mathbb{R}^{169} \mapsto \mathbb{R}$, which takes an islander's 169 attributes as an input and outputs their intelligence. Prove that there exists an intelligence function such that no islander has a higher intelligence than Bob.

Solution

The problem is equivalent to proving the existence of a function I such that $I(x) \leq I(B)$ for all possible $x = (x_1, x_2, \ldots, x_{169}) | x_k \in \mathbb{R}$.

Consider the function

$$I(x) = -\sum_{k=1}^{169} (x_k - k)^2$$

Clearly, as $(x_k - k)^2 \geq 0$, the function attains a maximum value when $x_k = k$ for all $1 \leq k \leq 169$. Thus $I(x) \leq I(B)$, as desired.

Problem 27A. Proposed by DC

In any triangle ABC, find the ratio

$$\frac{sinA + sinB + sinC}{cotA + cotB + cotC}$$

function of the altitudes in the triangle.

Problem 4B. Proposed by Ken Jiang

Alice and Bob are playing a new game. Starting from N, they take turns counting down F_i numbers, where F_i must be a member of the Fibonacci sequence. Alice goes first, and the player who counts to 1 is the winner. Show that there are infinite values of N such that, no matter how Alice plays, Bob can win.

Solution

We will prove the claim by contradiction. For a value to be winning for Alice, it must be either a Fibonacci number (she can count straight to 1), or it must be a Fibonacci number greater than a losing position. Contrarily, a losing position is one where counting any permitted value leads to a winning position.

Suppose there are a finite number of values of N where Alice loses. Then, there must exist a Fibonacci number F_n that is greater than all such values.

However, consider the numbers from $F_{n+2} + F_n$ to $F_{n+3} - 1$. Clearly, no Fibonacci number less than or equal to F_{n+2} can bring us to a losing position, since the resulting value would be greater than or equal to F_n, which we assumed was greater than any losing position. Obviously, we are also unable to count any Fibonacci number greater than or equal to F_{n+3}, since they are all greater than any of the values in this range. Thus, there are either losing positions between F_n and $F_{n+2} + F_n$, contradicting our original assumption, or all the values from $F_{n+2} + F_n$ to $F_{n+3} - 1$ are losing positions, also contradicting our original assumption.

Since our original assumption must lead to a contradiction, it must be false, which means there are an infinite number of values of N such that Bob can win.

Problem 5B. Proposed by Proposed by Alexander Monteith-Pistor

Let $S = \{0, 1, 2, ..., 2020\}$ and $f : S \to S$ satisfy

$$f(x)f(y)f(xy) = f(f(x + y))$$

for all $x, y \in S$ with $xy \leq 2020$ and $x + y \leq 2020$. Find the maximum possible value of

$$\sum_{i=0}^{2020} f(i)$$

Solution

We claim that the maximum value is $2020 \cdot 1975 = 3989500$. Throughout this proof, let

$$k := \sum_{i=0}^{2020} f(i)$$

Letting $y = 0$ we obtain $f(0)^2 f(x) = f(f(x))$ for all $x \in S$. Letting $y = 1$ (and $2020 \neq x \in A$) we obtain $f(1)f(x)^2 = f(f(x+1))$. Thus, $f(0)^2 f(x+1) = f(1)f(x)^2$ for all $2020 \neq x \in S$. We now consider two cases.

Case 1: $f(0) \neq 0$. Then

$$f(x) = \frac{f(1)}{f(0)^2} f(x-1)^2 \quad (*)$$

for all nonzero $x \in S$. Substituting $x = y = 2$ into the original equation,

$$f(2)^2 f(4) = f(f(4)) = f(0)^2 f(4)$$

Observe that if $f(4) = 0$, $f(x) = 0$ for all $x \geq 4$ (by induction with (*)). In this case, $k \leq 4 \cdot 2020$. Otherwise, $f(2) = f(0)$ (since both are positive). Using $x = 2$ in (*) we get $f(2) = f(1)$. Therefore

$$f(1) = \frac{f(1)}{f(0)^2} f(1)^2$$

it follows by induction with (*) that f is constant. Let $f(x) = c$ for all $x \in S$. Then the original equation yields $c^3 = c$ which implies $c = 0$ or $c = 1$. Thus, in this case, $k = 0$ or $k = 2021$.

Case 2: $f(0) = 0$. By the $y = 0$ substitution, $f(f(x)) = 0$ for all $x \in S$ which implies $f(x)f(y)f(xy) = 0$ for all $x, y \in S$ (satisfying $xy \leq 2020$). In any such case, we can partition S into two sets A and B where f maps each element of B to 0 and each element of A to a nonzero element of B. Further, if $xy \leq 2020$ then x, y, xy cannot all be in A.

We claim that A has at most 1976 elements. To prove this, let m be the minimum element in A (if A is empty the claim follows trivially). We have already determined that 0 and 1 are in B. If $m \geq 45$ then A has at most 1976 elements so assume $m \leq 44$. Then we can then create at least

$$p(m) := \left\lfloor \frac{2020}{m} \right\rfloor - \left\lfloor \frac{2020}{m^2} \right\rfloor \geq \left\lfloor \frac{2020(m-1)}{m^2} \right\rfloor - 1$$

pairs (n, nm) with distinct elements across all pairs. At least one element of each pair cannot be in A. Note $p(m) \geq 45$ for $1 \leq m \leq 42$. Thus, if $m \geq 43$ then A has at most 1975 elements (taking into account 0). If $m = 44$, $p(m) = 44$, $44^2 = 1936 \notin A$ therefore A has at most 1975 elements. This concludes the proof of the claim.

By the above casework, if A has 1976 elements, then $A = \{x \in S : x \geq 45\}$. f maps every element of A to an element of B therefore $k \leq 1976 \cdot 44 < 1975 \cdot 2020$. Finally, if A has less than 1976 elements $k \leq 1975 \cdot 2020$ (f is bounded above by 2020). Equality is achieved with the function

$$f(x) = \begin{cases} 0 & x \leq 44 \text{ or } x = 2020 \\ 2020 & 45 \leq x \leq 2019 \end{cases}$$

Problem 21B. Proposed by Alexander Monteith-Pistor

Let A_n denote the number of tuples $(a_1, ..., a_n)$ of positive integers which satisfy $a_1 = 1$ and $a_{k+1} \mid 2a_k$ for all $1 \leq k \leq n-1$ (note $A_0 = 1$). Prove that

$$A_{n+1} = \sum_{i=0}^{n} A_i A_{n-i}$$

Solution

For all $k \geq 0$, let S_k be the set of k-tuples satisfying the given conditions. For any $(a_1, ..., a_{n+1}) \in S_{n+1}$, let $f(a_1, ..., a_{n+1})$ be the maximum i such that $a_i = 1$. We can count the number of elements in S_{n+1} by counting the number of elements in S_{n+1} which f maps to i for each $i = 1, 2, ..., n, n+1$.

Let $f(a_1, ..., a_{n+1}) = i$ for some $1 \leq i \leq n+1$. Then $(a_1, a_2, ..., a_{i-1}) \in S_{i-1}$. Further, $\left(\frac{1}{2}a_{i+1}, .., \frac{1}{2}a_{n+1}\right) \in S_{n-i+1}$ ($a_{i+1} \mid 2$ and $a_{i+1} \neq 1$ so $a_{i+1} = 2$). Additionally, given any $(b_1, ..., b_{i-1}) \in S_{i-1}$ and $(b_{i+1}, ..., b_{n+1}) \in S_{n-i+1}$, $(b_1, ..., b_{i-1}, 1, 2b_{i+1}, ..., 2b_{n+1}) \in S_{n+1}$ with $f(b_1, ..., b_{i-1}, 1, 2b_{i+1}, ..., 2b_{n+1}) = i$. Thus, the number of elements of S_n which f maps to i is $A_{i-1}A_{n-i+1}$. As f maps every element of S_{n+1} to a unique element of $\{1, ..., n+1\}$,

$$A_n = \sum_{i=1}^{n+1} A_{i-1}A_{n-i+1} = \sum_{j=0}^{n} A_j A_{n-j}$$

Problem 22B. Proposed by Alexander Monteith-Pistor

Find the number of different black and white colourings of an $n \times n$ grid such that every square has exactly one black neighbour. Two squares are said to be neighbours if they share an edge.

Problem 23B. Proposed by Max Jiang

Mabel has a bag of n marbles whose weights are $1, 2, 3, \ldots, n$. She draws the marbles one by one without replacement out of the bag. At any point, the probably of drawing any given marble is proportional to its weight. For example, if the bag had 3 marbles of weights 1, 2, and 4, the probability of drawing the marble of weight 1, 2, and 4 are $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{4}{7}$, respectively. Find a closed form formula for the probability that Mabel draws the marbles in order of increasing weight.

Solution

Let marble k be the marble of weight k. We see that Mabel must draw marbles $1, 2, \ldots, n$ in that exact order.
The probability of drawing marble 1 first is $\frac{1}{\frac{n(n+1)}{2}}$. Then, the probability of drawing marble 2 after that is $\frac{2}{\frac{n(n+1)}{2} - \frac{1\cdot 2}{2}}$. In general, the probability of drawing marble k after drawing marbles $1, 2, \ldots, k-1$ is $\frac{k}{\frac{n(n+1)}{2} - \frac{(k-1)k}{2}}$. Thus, the probability that Mabel draws all the marbles in order of increasing weight is

$$\begin{aligned}
\prod_{i=1}^{n} \frac{i}{\frac{n(n+1)}{2} - \frac{(i-1)\cdot i}{2}} &= \prod_{i=1}^{n} \frac{i}{\frac{(n-(i-1))(n+i)}{2}} \\
&= \prod_{i=1}^{n} \frac{2i}{(n-(i-1))(n+i)} \\
&= 2^n n! \prod_{i=1}^{n} \frac{1}{n-(i-1)} \prod_{i=1}^{n} \frac{1}{n+i} \\
&= 2^n n! \prod_{i=1}^{n} \frac{1}{i} \prod_{i=1}^{n} \frac{1}{n+i} \\
&= \boxed{\frac{2^n n!}{(2n)!}}.
\end{aligned}$$

Problem 24B. Proposed by Vedaant Srivastava

Donnie holds 5 exclusive parties in order to increase his popularity among his 538 friends. At each party, some of his 538 friends were present and some were not. The ticket prices at each event were \$2, \$3, \$3, \$4, and \$5 respectively. Each event made over \$1000 in ticket sales. Prove that Donnie can choose two people out of his 538 friends such that at least one of them was present at each party.

Solution

Assume for contradiction that for each pair of people, there was at least one party in which both people were absent.

Let $\mathcal{S}$ be the number of pairs of people absent at the same party counted over all parties (ex : if two people are both absent in two different parties, this would contribute 2 cases to $\mathcal{S}$).
We establish contradictory bounds on $\mathcal{S}$ by counting in two ways.

As each party made over \$1000 in ticket sales, we have that there were at least 500, 334, 334, 250, and 200 people who attended each of the parties, respectively. This implies that there were at most 38, 204, 204, 288, and 338 people who were absent at each party. Choosing the maximum number of **pairs** of absent people at each party, we obtain that

$$\mathcal{S} \leq \binom{38}{2} + \binom{204}{2} + \binom{204}{2} + \binom{288}{2} + \binom{338}{2} = 140,396$$

However, for each pair of the 538 people, there was at least one party which both people did not attend. This implies that

$$\mathcal{S} \geq \binom{538}{2} = 144,453$$

Combining both bounds, we get

$$144,453 \leq \mathcal{S} \leq 140,396$$

which is clearly absurd. Thus by contradiction, there has to be some pair of people such that at least one of them was present at each party.

Problem 25B. Proposed by Andy Kim

Let

$$p(x) = x^5 + x^4 + ax^3 + bx^2 + cx + d$$

be a real polynomial with $1+i$ and $2+i$ as roots, where $i^2 = -1$. Find $a+b+c+d$.

Solution

Since p is a real polynomial, the complex conjugates of these roots must also be roots, so we have that $1+i, 1-i, 2+i, 2-i$ are roots. Let α be the fifth root. Then, noting that the coefficient of x^4 is 1, we have

$$\alpha + (1+i) + (1-i) + (2+i) + (2-i) = -1$$

and so

$$\alpha = -7$$

Then, we have all 5 roots, and thus

$$p(x) = (x+7)(x-(1+i))(x-(1-i))(x-(2+i))(x-(2-i))$$

and so

$$p(x) = (x+7)(x^2-2x+2)(x^2-4x+5)$$

Then, since $p(1) = 2 + a + b + c + d$, we have

$$\begin{aligned} a+b+c+d &= p(1) - 2 \\ &= (1+7)(1^2-2(1)+2)(1^2-4(1)+5) - 2 \\ &= (8)(1)(2) - 2 \\ &= 14 \end{aligned}$$

Problem 26B Proposed by Andy Kim

Alice and Bob each have a bucket with 1 liter of water in it to start. Every minute, Alice moves half of the water in her bucket to Bob's, and Bob moves a fourth of the water in his bucket to Alice's. Assuming Alice and Bob are immortal, find the limiting value of the amount of water in each bucket.

Solution

Let a_n and b_n be the amount of water left in Alice's and Bob's bucket respectively after n minutes, with $a_0 = b_0 = 1$.

Then, we have the following recurrence relation.

$$a_n = \frac{1}{2}a_{n-1} + \frac{1}{4}b_{n-1}$$
$$b_n = \frac{1}{2}a_{n-1} + \frac{3}{4}b_{n-1}$$

This can be represented with matrices as

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$$

Letting

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

we have

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

Then, we note that we have (can be found by diagonalization)

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

with

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so these two matrices are inverses. From this, we get

$$A^n = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}^n \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\frac{1}{4})^n \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

So, we have

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\frac{1}{4})^n \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

and we have the limiting values as n goes to infinity as

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$$

Problem 27B. Proposed by Nicholas Sullivan

Consider a continuous function $f(x)$ that satisfies $f(x)f(y) = f(x) + f(y) - f(xy)$, for any real numbers $x, y \neq 0$. If $f(2) = \frac{1}{2}$, then find all possible functions $f(x)$.

Solution

Since $f(x)f(y) = f(x) + f(y) - f(xy)$, then we know that

$$\begin{aligned} f(xy) &= f(x) + f(y) - f(x)f(y) \\ 1 - f(xy) &= (1 - f(x))(1 - f(y)). \end{aligned}$$

If we define $g(x) = 1 - f(x)$, then we have that

$$g(xy) = g(x)g(y).$$

Since this is true for all real numbers not equal to 0 or 1, then for any integer n,

$$g(x^n) = [g(x)]^n.$$

Similarly, we can show that for any rational q, and positive real x,

$$\begin{aligned} g(x^q) &= [g(x)]^q \\ g(-x^q) &= g(-1)[g(x)]^q. \end{aligned}$$

Thus, by continuity, we can say that for any positive real x,

$$\begin{aligned} g(x) &= g(e^{\ln x}) \\ &= [g(e)]^{\ln x} \\ &= x^{\ln g(e)} \\ g(-x) &= g(-1)g(e^{\ln x}) \\ &= g(-1)[g(e)]^{\ln x} \\ &= g(-1)x^{\ln g(e)} \end{aligned}$$

If we let $k = \ln g(e)$, then this means that $g(x) = x^k$ for some real number k. Thus, we can express $f(x)$ as

$$f(x) = \begin{cases} 1 - x^k & x > 0 \\ 1 - g(-1)(-x)^k & x < 0 \end{cases}.$$

The condition $f(2) = \frac{1}{2}$ implies that

$$\begin{aligned} f(2) &= 1 - 2^k \\ \frac{1}{2} &= 1 - 2^k \\ k &= -1. \end{aligned}$$

Thus, we can express $f(x)$ as

$$f(x) = \begin{cases} 1 - \frac{1}{x} & x > 0 \\ 1 - g(-1)\frac{1}{-x} & x < 0 \end{cases}.$$

Since $[g(-1)]^2 = g(1) = 1$, then $g(-1) = \pm 1$. This gives two options for $f(x)$.

If $g(-1) = 1$, then $f(x) = 1 - \frac{1}{|x|}$, and if $g(-1) = -1$, then $f(x) = 1 - \frac{1}{x}$.

Problem 28B. Proposed by DC

Find the relationship between a and b such that the maximum and the minimum of the function defined on real numbers:

$$\frac{x^2 + 2ax + 1}{x^2 + 2bx + 1}$$

are satisfying $\max = -2\min$.

Solution

Using the notation

$$\frac{x^2+2ax+1}{x^2+2bx+1}=m,$$

we obtain

$$(m-1)x^2+2(bm-a)x+m-1=0.$$

In order to have real roots, the discriminant Δ must be greater or equal to zero. Consequently,

$$\Delta=4[(bm-a)^2-(m-1)^2]=4[(b+1)m-a-1]\times[(b+1)m-a+1]\geq 0.$$

Considering the sign of the expressions $a-1$,$a+1$, $b-1$ and $b+1$ we have three out of four possible cases:
Case 1:

$$m\in(-\infty,\frac{a-1}{b-1}]\cup[\frac{a+1}{b+1},+\infty)$$

and

$$\frac{a-1}{b-1}=-2\frac{a+1}{b+1}.$$

$$(a-1)(b+1)=-2(a+1)(b-1)$$

and

$$ab+a-b-1=-2ab+2a-2b+2.$$

Finally

$$3ab-a+b-3=0$$

Case 2:

$$m\in[\frac{a+1}{b+1},\frac{a-1}{b-1}]$$

and

$$\frac{a-1}{b-1}=-2\frac{a+1}{b+1}.$$

With the same condition

$$3ab-a+b-3=0.$$

Case 3:

$$m\in(-\infty,\frac{a+1}{b+1}]\cup[\frac{a-1}{b-1},+\infty)$$

and

$$\frac{a+1}{b+1}=-2\frac{a-1}{b-1}.$$

$$(a+1)(b-1)=-2(a-1)(b+1)$$

and

$$ab-a+b-1=-2ab-2a+2b+2.$$

Finally

$$3ab + a - b - 3 = 0$$

Case 4:

$$m \in [\frac{a-1}{b-1}, \frac{a+1}{b+1}]$$

and

$$\frac{a+1}{b+1} = -2\frac{a-1}{b-1}.$$

With the same condition

$$3ab + a - b - 3 = 0.$$